ISRAEL JOURNAL OF MATHEMATICS, Vol. 41, Nos. 1-2, 1982

SPECTRA OF ERGODIC GROUP ACTIONS

BY

KLAUS SCHMIDT

ABSTRACT

Let G be a locally compact second countable abelian group, (X, μ) a σ-finite Lebesgue space, and $(g, x) \to gx$ a non-singular, properly ergodic action of G on (X, μ). Let furthermore Γ be the character group of G and let $\mathrm{Sp}(G, X) \subset \Gamma$ denote the L^∞-spectrum of G on (X, μ). It has been shown in [5] that $\mathrm{Sp}(G, X)$ is a Borel subgroup of Γ and that $\sigma(\mathrm{Sp}(G, X)) < 1$ for every probability measure σ on Γ with $\limsup_{g \to \infty} \mathrm{Re}\, \hat{\sigma}(g) < 1$, where $\hat{\sigma}$ is the Fourier transform of σ. In this note we prove the following converse: if σ is a probability measure on Γ with $\limsup_{g \to \infty} \mathrm{Re}\, \hat{\sigma}(g) = 1$ then there exists a non-singular, properly ergodic action of G on (X, μ) with $\sigma(\mathrm{Sp}(G, X)) = 1$.

Let G be a locally compact, second countable, non-compact abelian group, (X, μ) a σ-finite Lebesgue space, and $(g, x) \to gx$ a non-singular, properly ergodic action of G on (X, μ). An element $\gamma \in \Gamma$, the character group of G, is said to belong to the L^∞-spectrum $\mathrm{Sp}(G, X)$ of this action of G if there exists a non-zero function $f \in L^\infty(X, \mu)$ with $f(gx) = \gamma(g) f(x)$ μ-a.e., for every $g \in G$. $\mathrm{Sp}(G, X)$ is clearly a subgroup of Γ, and it is not difficult to verify that it is also a Borel set. It is furthermore known that $\lambda_\Gamma(\mathrm{Sp}(G, X)) = 0$, where λ_Γ is the Haar-measure on Γ, but that $\mathrm{Sp}(G, X)$ may be uncountable. The following result gives a complete metric characterization of the possible subgroups which may occur as $\mathrm{Sp}(G, X)$.

THEOREM. *Let G be a locally compact, second countable, non-compact, abelian group with character group Γ, and let σ be a probability measure on Γ with Fourier transform $\hat{\sigma}$. The following conditions are equivalent.*

(1) *There exists a non-singular, properly ergodic action $(g, x) \to gx$ of G on a Lebesgue space (X, μ) such that $\sigma(\mathrm{Sp}(G, X)) = 1$,*

(2) $\limsup_{g \to \infty} \mathrm{Re}\, \hat{\sigma}(g) = 1$, *where* Re *denotes the real part.*

PROOF. The implication (1) $\Rightarrow$ (2) is an immediate consequence of theorem 4.1 in [5]. To prove the converse, let σ be a probability measure on Γ satisfying

Received November 5, 1980 and in revised form May 30, 1981

(2), and let σ' be a purely atomic probability measure on Γ with $\sigma'(\{1\})>0$, and whose set of atoms is dense in Γ. Put $\tau=\sigma*\sigma'$, where * denotes convolution. An elementary argument shows that $\limsup_{g\to\infty}\operatorname{Re}\hat{\sigma}(g)=1$ (for an explicit proof see lemma 3.4 in [7]). Furthermore, if $B\subset\Gamma$ is a Borel set with $\tau(B)=1$, then we clearly have $\sigma(B)=1$. It will thus be sufficient to construct a Lebesgue space (X,μ) and a non-singular, properly ergodic action $(g,x)\to gx$ of G on (X,μ) with $\tau(\operatorname{Sp}(G,X))=1$. Let $X=\mathcal{U}(\Gamma,\tau,S^1)$ be the group of τ-equivalence classes of Borel maps from Γ to S^1, the multiplicative group of complex numbers of modulus 1, under pointwise multiplication and furnished with the metric

$$d(x_1,x_2)=\int|x_1(\gamma)-x_2(\gamma)|\,d\tau(\gamma),\qquad x_1,x_2\in X.$$

(X,d) is a complete, separable metric group, and G acts continuously and isometrically on X by $gx(\gamma)=\gamma(g)x(\gamma)$ for every $x\in X$ and $g\in G$. This action of G on X is free (i.e. $gx\neq x$ whenever $g\neq e$), and, in particular, the map $g\to g.1=x_g$ from G into X is injective (here 1 denotes the constant function 1 in X). From $\limsup_{g\to\infty}\operatorname{Re}\hat{\tau}(g)=1$ we conclude that $\liminf_{g\to\infty}d(x_e,x_g)=0$, so that the map $g\to x_g$ is not a homeomorphism onto its image. By [1, theorem 1 and 4, theorem 2.6] there exists a probability measure μ on X which is quasi-invariant and ergodic under G and which is not concentrated on a single G-orbit (for a more detailed discussion of this phenomenon and its applications we refer to [8]). In other words, the action $(g,x)\to gx$ of G on (X,μ) is non-singular and properly ergodic Consider now the map $F:\Gamma\times X\to S^1$ given by $F(\gamma,x)=x(\gamma)$. By [9, lemma 8.5] F is well-defined $\tau\times\mu$-a.e. and may be chosen to be a Borel map. Since $F(\gamma,gx)=\gamma(g)F(\gamma,x)$ $\tau\times\mu$-a.e., for every $g\in G$, we can use Fubini's theorem to find a Borel set $S\subset\Gamma$ with $\tau(S)=1$, and such that $F(\gamma,gx)=\gamma(g)F(\gamma,x)$ $\lambda_G\times\mu$-a.e., for every $\gamma\in S$, where λ_G denotes the Haar measure of G. Applying once again Fubini's theorem we can find, for every $\gamma\in S$, a Borel set $X_\gamma\subset X$ with $\mu(X_\gamma)=1$ and with $F(\gamma,gx)=\gamma(g)F(\gamma,x)$ λ_G-a.e., for every $x\in X_\gamma$. A standard argument now shows that $F(\gamma,.)$ is an eigenfunction with eigenvalue γ, for every $\gamma\in S$. This in turn implies that $S\subset\operatorname{Sp}(G,X)$, so that $\sigma(\operatorname{Sp}(G,X))=\tau(\operatorname{Sp}(G,X))=1$. □

REMARK 1. The following is an interesting consequence of the above theorem. Let (X,μ) be a Lebesgue space, and let σ be a probability measure on R (the real line) with $\sigma(\{0\})=0$ and $\limsup_{s\to\infty}\operatorname{Re}\hat{\sigma}(s)=1$. As we have shown, there exists a non-singular, properly ergodic flow $(T_t:t\in R)$ with $\sigma(\operatorname{Sp}(T_t,t\in R))=1$. Clearly, if s lies in the L^∞-spectrum of $(T_t,t\in R)$, then $T_{1/s}$

is non-ergodic. In particular we conclude that $\sigma(\{s \in R : s \neq 0$ and $T_{1/s}$ is non-ergodic$\}) = 1$. This is related to problems discussed in [5].

REMARK 2. Using the methods of [2, 6] one can show that the G-action on (X, μ) in the above theorem may be chosen to preserve an infinite, σ-finite measure equivalent to μ.

REMARK 3. In the special case $G = Z$ (the group of integers), B. Weiss has a much more direct (unpublished) proof of the implication (2) $\Rightarrow$ (1) in the above theorem. The method used in our proof is related to an idea by E. Flytzanis in [3] and to techniques used in [8].

REMARK 4. For every locally compact, second countable, non-discrete, abelian group Γ with character group G there exists a non-atomic probability measure σ on Γ satisfying $\limsup_{g \to \infty} \operatorname{Re} \hat{\sigma}(g) = 1$ (cf. [7]). An explicit method for constructing such measures on S^1 is described in [8, §5].

REFERENCES

1. E. G. Effros, *Transformation groups and C*-algebras*, Math. Ann. **81** (1965), 38–55.

2. J. Feldman, P. Hahn and C. C. Moore, *Orbit structure and countable sections for actions of continuous groups*, Advances in Math. **28** (1978), 186–230.

3. E. Flytzanis, *Rigid factors of ergodic transformations*, **35** (1980), 323–326.

4. J. Glimm, *Locally compact transformation groups*, Trans. Amer. Math. Soc. **101** (1961), 124–138.

5. V. Losert and K. Schmidt, *A class of probability measures arising from some problems in ergodic theory*, in *Probability Measures on Groups*, Springer Lecture Notes in Mathematics, Vol. 706, 1979, pp. 220–238.

6. K. Schmidt, *Infinite invariant measures on the circle*, Symposia Mathematica **XXI** (1977), 37–43.

7. K. Schmidt, *Mildly mixing actions of locally compact groups. II*, preprint, Warwick, 1981.

8. K. Schmidt and P. Walters, *Mildly mixing actions of locally compact groups. I*, preprint, Warwick, 1981.

9. V. S. Varadarajan, *Geometry of Quantum Theory. II*, Van Nostrand, 1970.

MATHEMATICS INSTITUTE
UNIVERSITY OF WARWICK
COVENTRY CV4 7AL, UK